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AUTHOR(S):

Artbazar, Galtbayar; Jensen, Arne

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The Nelson Model with Few Photons

Galtbayar Artbazar — University of Tokyo
Arne Jensen — Aalborg University, Denmark
Kenji Yajima (谷島賢二) — University of Tokyo

1 Introduction

In his celebrated paper, Nelson proposed a simplified model of nonrelativisitic quantum electrodynamics in which interaction between a matter and the field is given as a linear function of the field operator. Thus, to describe an atom interacting with a (scalar) radiation field, he proposed to study the Hamiltonian given by

$$H_{\text{Nelson}} = -\tfrac{1}{2}\Delta_x + V(x) + \int_{\mathbf{R}^3} \omega(k)a^\dagger(k)a(k)dk + \Phi(x)$$

which acts on the state space defined by $\mathcal{H}_{\text{Nelson}} = L^2(\mathbf{R}^3_x) \otimes \mathcal{F}$. Here

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbf{R}^3_k)$$

is the boson Fock space, $\otimes_s^n$ being the n-fold symmetric tensor product with $\otimes_s^0 L^2(\mathbf{R}^3) \equiv \mathbf{C}$; $-\frac{1}{2}\Delta_x + V(x)$, in $L^2(\mathbf{R}^3_x)$ is the electron Hamiltonian, where V is the decaying real potential describing the interaction between the electron and the nucleus; $a(k)$ and $a^\dagger(k)$ are, respectively, the the annihilation and the creation operator; $\omega(k) = |k|$ is the dispersion relation and

$$\int_{\mathbf{R}^3} \omega(k)a^\dagger(k)a(k)dk$$

is the photon energy operator; and the interaction between the field and the electron is given by

$$\Phi(x) = \mu \int_{\mathbf{R}^3} \frac{\chi(k)}{\sqrt{\omega(k)}} \left\{e^{-ikx}a^\dagger(k) + e^{ikx}a(k)\right\} dk.$$

where $\mu > 0$ is the coupling constant and $\chi(k)$ is the ultraviolet cut-off function which we assume to satisfy the following assumption, where we use the standard notation $\langle k\rangle = (1 + k^2)^{1/2}$.

Assumption 1.1. *The function $\chi(k)$ is positive, smooth, $O(3)$-invariant and monotonically decreasing as $|k| \to \infty$. Moreover, $|\chi(k)| \leq C\langle k\rangle^{-N}$ for a sufficiently large N.*

In this paper, we study the restriction of this model to the subspace with less than two photons: Let P denote the projection onto the subspace of $\mathcal{H}_{\mathrm{Nelson}}$ given by

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad \mathcal{H}_0 = L^2(\mathbf{R}^3_x), \quad \mathcal{H}_1 = L^2(\mathbf{R}^3_x) \otimes L^2(\mathbf{R}^3_k),$$

which consists of states with less than two photons. Then we consider the Hamiltonian $H = PH_{\mathrm{Nelson}}P$ on this space. With respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ this Hamiltonian has the following matrix representation

$$H = \begin{pmatrix} -\frac{1}{2}\Delta + V & \mu\langle g| \\ \mu|g\rangle & -\frac{1}{2}\Delta + V + \omega(k) \end{pmatrix}.$$

Here we have defined the operators $|g\rangle\colon \mathcal{H}_0 \to \mathcal{H}_1$ and $\langle g|\colon \mathcal{H}_1 \to \mathcal{H}_0$ by

$$(|g\rangle u_0)(x,k) = g(x,k)u_0(x), \qquad (\langle g|u_1)(x) = \int_{\mathbf{R}^3} \overline{g(x,k)}u_1(x,k)dk,$$

where the function $g(x,k)$ is given by

$$g(x,k) = \frac{\chi(k)e^{-ixk}}{\sqrt{\omega(k)}}. \tag{1.1}$$

We write $g_0(k) = |g(x,k)| = \chi(k)/\sqrt{\omega(k)}$. It is obvious that $|g\rangle$ is bounded from $\mathcal{H}_0$ to $\mathcal{H}_1$, and that $\langle g|$ is its adjoint. We assume that V is $-\Delta$-compact so that H is a selfadjoint operator with the domain

$$D(H) = H^2(\mathbf{R}^3) \oplus \left(H^2(\mathbf{R}^3) \otimes L^2(\mathbf{R}^3) \cap L^2(\mathbf{R}^3) \otimes L^2_1(\mathbf{R}^3)\right).$$

Here $L^2_1(\mathbf{R}^3)$ denotes the usual weighted L^2-space, given by

$$L^2_1(\mathbf{R}^3) = L^2(\mathbf{R}^3, \langle k\rangle^2 dk),$$

and $H^2(\mathbf{R}^3)$ is the Sobolev space of order 2. We denote by H_0 the operator H with $V \equiv 0$. H_0 is the Hamiltonian for free electron-photon system and is translation invariant. Our goal is to describe the dynamics of this model. To state main results obtained, we need introduce the function

$$F(\xi,\lambda) = \tfrac{1}{2}\xi^2 - \lambda - \int \frac{\mu^2 |g_0(k)|^2 dk}{\frac{1}{2}(\xi - k)^2 + \omega(k) - \lambda} \tag{1.2}$$

defined on $\Gamma^- = \{(\xi,\lambda) : \lambda < \lambda_c(\xi) = \min_{k \in \mathbf{R}^3} \frac{1}{2}(\xi - k)^2 + \omega(k)\}$. $F(\xi,\lambda)$ is real analytic on Γ^-. We shall show in Section 2 that there exists a threshold momentum $\rho_c > 1$ such that the equation $F(\xi,\lambda) = 0$ for λ has a unique solution $\lambda_\circ(\xi) \in \mathbf{R}$ when $|\xi| \le \rho_c$ and no zeros when $|\xi| > \rho_c$. $\lambda_\circ(\xi)$ is $O(3)$-invariant, real analytic, strictly increasing with respect to $\rho = |\xi|$ and $\lambda_{\circ\rho\rho}(\rho) > 0$ for $\rho < \rho_c$. In what follows $\hat{u}$ stands for the Fourier transform of u with respect to the x variables.

Theorem 1.2. *For any* $\mathbf{f} \in \mathcal{H}$, *there uniquely exist* $\mathbf{f}_1 = \begin{pmatrix} f_{1,0} \\ f_{1,1} \end{pmatrix} \in \mathcal{H}$ *and* $f_{2,1,\pm} \in \mathcal{H}_1$ *such that, as* $t \to \pm\infty$,

$$\left\| e^{-itH_0}\mathbf{f} - \begin{pmatrix} e^{-it\lambda_\circ(D)} f_{1,0} \\ e^{-ikx} e^{-it\lambda_\circ(D_x)} f_{1,1} \end{pmatrix} - \begin{pmatrix} 0 \\ e^{it\Delta/2 - it\omega(k)} f_{2,1,\pm} \end{pmatrix} \right\| \to 0.$$

Here $\mathbf{f}_1$ *and* $f_{2,1,\pm}$ *satisfy the following properties:*

(1) $\hat{f}_{1,0}(\xi)$ *is supported by* $B(\rho_c) \equiv \{\xi : |\xi| < \rho_c\}$.

(2) $\hat{f}_{1,1}(\xi,k) = \mu g_0(k) \hat{f}_{1,0}(\xi) / (\frac{1}{2}(\xi - k)^2 + \omega(k) - \lambda_\circ(\xi))$.

(3) *The map* $\mathbf{f} \mapsto \begin{pmatrix} \hat{f}_{1,0} \\ \hat{f}_{2,1,\pm} \end{pmatrix}$ *is one to one and onto* $L^2(B(\rho_c)) \oplus L^2(\mathbf{R}^6)$.

(4) $\|\mathbf{f}\|^2_{\mathcal{H}} = \|\mathbf{f}_1\|^2_{\mathcal{H}} + \|f_{2,1,\pm}\|^2_{L^2(\mathbf{R}^6)}$.

This result shows, in particular, electron with large momentum $|\xi| > \rho_c$ in the vacuum state does not survive. One might associate this phenomenon to Cherenkov radiation, in the sense that the electron of high speed always carries one photon. However, it is not clear how relevant this description is. Usually Cherenkov radiation is described differently, in a classical electrodynamic context, see for example [2].

Theorem 1.3. *Let $V \in L^2(\mathbf{R}^3)$. Then, the wave operators*

$$W_{0\pm} = \underset{t\to\pm\infty}{s\text{-}\lim}\, e^{itH}e^{-itH_0} \tag{1.3}$$

exist on $\mathcal{H}$.

In the following theorem we can even allow V to be a typical N-body potential. Note that while eigenfunctions usually decay exponentially, those embedded at thresholds may decay only polynomially.

Theorem 1.4. *Assume that V is bounded relative to H_0 with bound less than one. Assume that E is an eigenvalue of $-\frac{1}{2}\Delta + V$ with normalized eigenfunction Ω. Assume there exist $C > 0$ and $\beta > 5/2$, such that $|\Omega(x)| \leq C\langle x\rangle^{-\beta}$ for $x \in \mathbf{R}^3$. Then for $f \in L^2(\mathbf{R}^3_k)$ the following limits exist.*

$$W_\pm^{E,\Omega} f = \lim_{t\to\pm\infty} e^{itH}\begin{pmatrix} 0 \\ e^{-itE-it(\omega(k)}\Omega(x)f(k)\end{pmatrix}. \tag{1.4}$$

Concerning the literature on this problem, then there seems to be no papers describing the asymptotics of our Hamiltonian in the manner done here. There is a large number of papers studying the Nelson model, when the atom is modelled by either $-\frac{1}{2}\Delta + V$ with compact resolvent (confining potential), or when the atom in modelled by a finite level system (spin-Boson Hamiltonians). The Nelson model was introduced in [4]. A detailed study of the case of atomic (or matter) Hamiltonians with compact resolvent and photons with $m > 0$ is given in [1].

Finally, let us outline the contents of this paper. In §2 we study in detail the properties of the function $F(\xi, z)$. In §3 where we determine the spectrum of H_0 by separating the center of mass motion. In §4 we study the operator e^{-itH_0} and prove Theorem 1.3. In §5 we prove Theorem 1.4. In §6 we prove Theorem 1.3.

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2 Properties of the function $F(\xi, z)$

As a first step we study the following function $F(\xi, z)$, defined for $(\xi, z) \in \mathbf{R}^3 \times (\mathbf{C} \setminus [m, \infty))$ by

$$F(\xi, z) = \tfrac{1}{2}\xi^2 - z - \int \frac{\mu^2 |g_0(k)|^2 dk}{\frac{1}{2}(\xi - k)^2 + \omega(k) - z}. \tag{2.1}$$

It plays a crucial role, since it enters into the resolvent of H_0 given in §3 and its zeros define the eigenvalues of reduced operators when the center of mass motion is removed from H_0. We study the properties of $F(\xi, z)$ in this section. The following Lemma is obvious.

Lemma 2.1. (1) *For each z the function $F(\xi, z)$ is $O(3)$-invariant.*
(2) $\mp \operatorname{Im} F(\xi, z) > 0$, *when* $\pm \operatorname{Im} z > 0$.
(3) *Let $K \subset \mathbf{C} \setminus [m, \infty)$ be a compact set. Then we have that $|F(\xi, z)| \to \infty$ as $|\xi| \to \infty$, uniformly with respect to $z \in K$.*

We will write $F(\rho, z) = F(\xi, z)$, $\rho = |\xi|$, and $F_\rho(\rho, z)$ will denote the derivative of $F(\rho, z)$ with respect to ρ. We will study the boundary values of $F(\xi, z)$ as $z = \lambda \pm i\varepsilon \to \lambda \in \mathbf{R}$, for $\varepsilon \downarrow 0$, and also other properties of this function.

We start by investigating the denominator in the integral. Let $G(\xi, k) = \frac{1}{2}(\xi - k)^2 + \omega(k)$. Then elementary computations show that for each fixed ξ the function $k \to G(\xi, k)$ has a global minimum, which we denote by $\lambda_c(\xi)$. Due to the invariance it is only a function of ρ. We have

$$\lambda_c(\rho) = \begin{cases} \frac{1}{2}\rho^2 & \text{for } 0 \le \rho \le 1, \\ \rho - \frac{1}{2} & \text{for } 1 < \rho. \end{cases} \tag{2.2}$$

Note that this function is only once continuously differentiable.

Denote by γ the curve in the right half plane given by

$$\gamma = \{(\rho, \lambda_c(\rho)) \colon \rho \ge 0\} \subset \{(\rho, \lambda) \colon \rho \ge 0, -\infty < \lambda < \infty\}. \tag{2.3}$$

Denote by $\Gamma^\pm$ the regions below and above γ:

$$\Gamma^- = \{(\rho, \lambda) \colon \rho \ge 0, \lambda < \lambda_c(\rho)\}, \qquad \Gamma^+ = \{(\rho, \lambda) \colon \rho \ge 0, \lambda > \lambda_c(\rho)\}.$$

We denote by the same symbols γ and $\Gamma^\pm$ the surface, and the domains, defined by

$$\gamma = \{(\xi, \lambda_c(\xi)) \colon \xi \in \mathbf{R}^3\} \subset \mathbf{R}^3 \times \mathbf{R},$$

$$\Gamma^- = \{(\xi,\lambda)\colon \xi \in \mathbf{R}^3, \lambda < \lambda_c(\xi)\}, \qquad \Gamma^+ = \{(\xi,\lambda)\colon \xi \in \mathbf{R}^3, \lambda > \lambda_c(\xi)\}.$$

Because of the $O(3)$-invariance of the functions used in the definitions, the double use of these symbols should not cause any confusion.

2.1 Zeros of $F(\rho, \lambda)$ in Γ^-

It is obvious that, on Γ^-, the function $F(\rho, \lambda)$ is real analytic with respect to λ.

Lemma 2.2. *In Γ^- the function $F(\rho, \lambda)$ is strictly decreasing with respect to λ and is strictly increasing with respect to ρ.*

Proof. We show that the derivatives satisfy $F_\lambda(\rho, \lambda) < 0$, and $F_\rho(\rho, \lambda) > 0$ in Γ^-. Direct computation shows

$$\frac{\partial F}{\partial \lambda} = -1 - \int \frac{\mu^2 |g(_0k)|^2 dk}{(\frac{1}{2}(\xi - k)^2 + \omega(k) - \lambda)^2} < 0.$$

To prove $F_\rho > 0$, it suffices to show that $F_{\xi_1}(\xi, \lambda) > 0$, when $\xi_1 \geq 0, \xi_2 = \xi_3 = 0$, as F is $O(3)$-invariant. We compute

$$\frac{\partial F}{\partial \xi_1} = \xi_1 + \int \frac{\mu^2 |g_0(k)|^2 (\xi_1 - k_1) dk}{(\frac{1}{2}(\xi - k)^2 + \omega(k) - \lambda)^2} = \xi_1 - \int \frac{\mu^2 |g_0(\xi + k)|^2 k_1 dk}{(\frac{1}{2}k^2 + \omega(\xi + k) - \lambda)^2}.$$

The last integral can be written in the form

$$\mu^2 \int_{\mathbf{R}^2} \left\{ \int_0^\infty \left(\frac{|g_0(\xi_1 - k_1, k')|^2}{(\frac{1}{2}k^2 + \omega(\xi_1 - k_1, k') - \lambda)^2} - \frac{|g_0(\xi + k)|^2}{(\frac{1}{2}k^2 + \omega(\xi + k) - \lambda)^2} \right) k_1 dk_1 \right\} dk',$$

where $k' = (k_2, k_3) \in \mathbf{R}^2$ and, for $\xi_1, k_1 > 0$,

$$|g_0(\xi_1 - k_1, k')|^2 > |g_0(\xi_1 + k_1, k')|^2,$$

$$\sqrt{(\xi_1 - k_1)^2 + (k')^2 + m^2} \leq \sqrt{(\xi_1 + k_1)^2 + (k')^2 + m^2}.$$

Here the first inequality follows from Assumption 1.1. Thus the integral is positive, and the lemma follows. □

Remark 2.3. *Computation by using polar coordinates yields for $(\rho, \lambda) \in \Gamma^-$*

$$F(\xi, \lambda) = \tfrac{1}{2}\rho^2 - \lambda - \frac{2\pi\mu^2}{\rho} \int_0^\infty |g_0(r)|^2 r \log\left(1 + \frac{4\rho r}{\frac{1}{2}(r-\rho)^2 + \omega(r) - \lambda}\right) dr. \quad (2.4)$$

Lemma 2.4. *There exist a constant $\rho_c > 0$ and a function $\lambda_\circ \colon [0, \rho_c] \to \mathbf{R}$ with the following properties:*
(i) $\lambda_\circ(0) < 0$, $\lambda_\circ(\rho_c) \in \gamma$, *and*

$$\Xi = \{(\rho, \lambda_\circ(\rho)) \colon 0 \le \rho \le \rho_c\} \subset \Gamma^- \cup \gamma. \quad (2.5)$$

(ii) $F(\rho, \lambda_\circ(\rho)) = 0$, $\rho \in [0, \rho_c]$.
(iii) $\lambda_\circ$ *is real analytic.*
(iv) $\lambda_{\circ\rho}(\rho) > 0$ *and* $\lambda_{\circ\rho\rho}(\rho) > 0$ *for* $0 < \rho < \rho_c$.
(v) *There are no other zeros of* $F(\rho, \lambda)$ *in* Γ^-, *than those given by* Ξ *in* (2.5).

Proof. We have (recall (2.2))

$$F(\rho, \lambda_c(\rho)) = -\int \frac{\mu^2 |g_0(k)|^2 dk}{\frac{1}{2}k^2 - \xi \cdot k + |k|} < 0$$

for $\rho \le 1$, and it is increasing for $\rho > 1$ and diverges to ∞ as $\rho \to \infty$. Indeed, we have

$$F(\rho, \lambda_c(\rho)) = \tfrac{1}{2}(\rho - 1)^2 - \frac{\pi\mu^2}{2\rho} \int_0^\infty |\chi(r)|^2 \log\left(1 + \frac{4\rho r}{(r - \rho + 1)^2}\right) dr$$

for $\rho > 1$ and it is evident that $\lim_{\rho \to \infty} F(\rho, \lambda_c(\rho)) = \infty$. By a change of variable,

$$\begin{aligned}
F(\rho + 1,&\lambda_c(\rho + 1)) \\
&= \rho^2 - \frac{\pi\mu^2}{2(\rho+1)} \int_0^\infty |\chi(r)|^2 \log\left(1 + \frac{4r(\rho+1)}{(r-\rho)^2}\right) dr \\
&= \rho^2 - \frac{\pi\mu^2\rho}{2(\rho+1)} \int_0^\infty |\chi(\rho r)|^2 \log\left(1 + \frac{4r(1+\frac{1}{\rho})}{(r-1)^2}\right) dr \\
&= \rho\Big[\rho - \frac{\pi\mu^2}{2(\rho+1)} \int_0^\infty |\chi(\rho r)|^2 \log\left(1 + \frac{4r(1+\frac{1}{\rho})}{(r-1)^2}\right) dr\Big].
\end{aligned}$$

This is manifestly increasing for $\rho > 0$. Thus, there exists a unique $\rho_c > 1$ such that $F(\rho, \lambda_c(\rho))$ changes sign from $-$ to $+$ at $\rho = \rho_c$. It follows, since $F(\rho, \lambda)$ in Γ^- is decreasing with respect to λ and $F(\xi, \lambda) \to \infty$ as $\lambda \to -\infty$ that the function $\lambda \to F(\rho, \lambda)$ has a unique zero $\lambda_\circ(\rho)$ for $0 \le \rho \le \rho_c$ and $\lambda_\circ(0) < 0$. By the implicit function theorem, $\lambda_\circ(\rho)$ is real analytic, and $\lambda_{\circ\rho}(\rho) > 0$ for $\rho < \rho_c$. Using implicit differentiation to find the second derivative, we get

$$\begin{aligned}-F_\lambda(\rho, \lambda_\circ(\rho))\lambda_{\circ\rho\rho}(\rho) &= F_{\lambda\lambda}(\rho, \lambda_\circ(\rho))(\lambda_{\circ\rho}(\rho))^2 \\ &\quad + 2F_{\lambda\rho}(\rho, \lambda_\circ(\rho))\lambda_{\circ\rho}(\rho) + F_{\rho\rho}(\rho, \lambda_\circ(\rho)).\end{aligned}$$

Using computations similar to those in the proof of Lemma 2.2, one can show that $F_{\lambda\lambda}(\rho, \lambda) > 0$, $F_{\lambda\rho}(\rho, \lambda) \ge 0$, and $F_{\rho\rho}(\rho, \lambda) \ge 0$ for $(\rho, \lambda) \in \Gamma^-$. The details are omitted. Now using $F_\lambda(\rho, \lambda) < 0$ in Γ^-, the statement (iv) follows. □

As above, we will also consider $\lambda_\circ$ as a function of ξ, through $\rho = |\xi|$. The Hessian of $\xi \to \lambda_\circ(\xi)$ is given by

$$\nabla_\xi^2 \lambda_\circ(\rho) = \lambda_{\circ\rho\rho}(\rho)\hat{\xi} \otimes \hat{\xi} + \lambda_{\circ\rho}(\rho)\frac{1 - (\hat{\xi} \otimes \hat{\xi})}{|\xi|}.$$

It follows from Lemma 2.4(iv) that $\nabla_\xi^2 \lambda_\circ(\rho)$ is strictly positive.

3 Spectrum and resolvent of H_0

In this section we study the case when $V = 0$ and define

$$H_0 = \begin{pmatrix} -\frac{1}{2}\Delta & \mu\langle g| \\ \mu|g\rangle & -\frac{1}{2}\Delta + \omega(k) \end{pmatrix}.$$

3.1 Separation of the center of mass

It is easy to see that the operator H_0 commutes with the spatial translations

$$\tau_j(s) \ : \ \begin{pmatrix} u_0(x) \\ u_1(x,k) \end{pmatrix} \mapsto \begin{pmatrix} u_0(x + se_j) \\ e^{isk_j} u_1(x + se_j, k) \end{pmatrix}, \quad s \in \mathbf{R}, \ j = 1, 2, 3.$$

Hence H_0 and the generators $P_j = \begin{pmatrix} -i\partial/\partial x_j & 0 \\ 0 & -i\partial/\partial x_j + k_j \end{pmatrix}$ of $\tau_j(s)$ can simultaneously be diagonalized. We let $\mathcal{K} = \mathbf{C} \oplus L^2(\mathbf{R}^3)$ and define the unitary operator $U : \mathcal{H} \to L^2(\mathbf{R}^3_p : \mathcal{K}) = \int_{\mathbf{R}^3}^{\oplus} \mathcal{K} dp$ by

$$U : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{u}_0(p) \\ \tilde{u}_1(p,k) \end{pmatrix} = \begin{pmatrix} \hat{u}_0(p) \\ \hat{u}_1(p-k,k) \end{pmatrix}.$$

With respect to this direct decomposition, we have

$$UH_0U^* = \int_{\mathbf{R}^3}^{\oplus} H_0(p)dp, \quad H_0(p) = \begin{pmatrix} \frac{1}{2}p^2 & \mu\langle g_0| \\ \mu|g_0\rangle & \frac{1}{2}(p-k)^2 + \omega(k) \end{pmatrix} \tag{3.1}$$

where $\langle g_0| : L^2(\mathbf{R}^3) \ni \tilde{u}_1 \mapsto (\tilde{u}_1, g_0)_{L^2} \in \mathbf{C}$ and $|g_0\rangle : \mathbf{C} \ni c \mapsto cg_0(k) \in L^2(\mathbf{R}^3)$ are operators of rank one. Thus,

$$H_0(p) = \begin{pmatrix} \frac{1}{2}p^2 & 0 \\ 0 & \frac{1}{2}(p-k)^2 + \omega(k) \end{pmatrix} + \begin{pmatrix} 0 & \mu\langle g_0| \\ \mu|g_0\rangle & 0 \end{pmatrix} = H_{00}(p) + T$$

is the rank two perturbation of $H_{00}(p)$. $H_0(p)$ is essentially the operator known as Friedrichs model. Thus, it is standard to compute its resolvent and, if we write

$$(H_0(p) - z)^{-1}\tilde{\mathbf{f}} = \begin{pmatrix} \tilde{u}_0(p,z), \\ \tilde{u}_1(p,k,z) \end{pmatrix}, \quad \tilde{\mathbf{f}} = \begin{pmatrix} \tilde{f}_0, \\ \tilde{f}_1(k) \end{pmatrix}, \tag{3.2}$$

we have

$$\tilde{u}_0(p,z) = \frac{1}{F(p,z)} \left(\tilde{f}_0 - \mu \int \frac{g_0(k)\tilde{f}_1(k)dk}{\frac{1}{2}(p-k)^2 + \omega(k) - z} \right), \tag{3.3}$$

$$\tilde{u}_1(p,k,z) = \frac{\tilde{f}_1(k)}{\frac{1}{2}(p-k)^2 + \omega(k) - z} - \frac{\mu g_0(k)\tilde{u}_0(p,z)}{\frac{1}{2}(p-k)^2 + \omega(k) - z}. \tag{3.4}$$

3.2 Spectrum of the reduced operators

Theorem 3.1. (1) *When $|p| < \rho_c$, the spectrum $\sigma(H_0(p))$ of $H_0(p)$ consists of simple eigenvalue $\lambda_\circ(p)$ and the absolute continuous part $[\lambda_c(p), \infty)$. The* *normalized eigenfunction associated with the eigenvalue $\lambda_\circ(p)$ may be given by*

$$\mathbf{e}(p) = \frac{1}{\sqrt{-F_\lambda(p,\lambda_\circ(p))}} \begin{pmatrix} 1 \\ \dfrac{\mu g_0(k)}{\frac{1}{2}(p-k)^2+\omega(k)-\lambda_\circ(p)} \end{pmatrix}. \tag{3.5}$$

(2) *When $|p| \geq \rho_c$, $\sigma(H_0(p)) = [\lambda_c(p), \infty)$ and is absolute continuous.*

Proof. As was shown in Lemma 2.2, $F(p,z)$ is an analytic function of $z \in \mathbf{C}\backslash[\lambda_c(p),\infty)$, it has a simple zero at $\lambda_\circ(p)$ when $|p| < \rho_c$ and has no zero when $|p| \geq \rho_c$. It follows from (3.2) $\sim$ (3.4) that $\mathbf{C}\backslash[\lambda_c(p),\infty) \ni z \mapsto (H_0(p)-z)^{-1}$ is meromorphic with a simple pole $\lambda_\circ(p)$ if $|p| < \rho_c$, and it is holomorphic if $|p| \geq \rho_c$. Hence:

1. If $|p| < \rho_c$, $H_0(p)$ has an eigenvalue $\lambda_\circ(p)$ and $(-\infty, \lambda_c(p)) \setminus \{\lambda_\circ(p)\} \subset \rho(H_0(p))$, $\rho(H_0(p))$ being the resolvent set of $H_0(p)$.
2. If $|p| \geq \rho_c$, $(-\infty, \lambda_c(p)) \subset \rho(H_0(p))$.

By virtue of (3.2) $\sim$ (3.4), we can compute the eigenprojection E_p for $H_0(p)$ associated with the eigenvalue $\lambda_\circ(p)$ as follows:

$$E_p = -\lim_{z\to\lambda_\circ(p)} (z-\lambda_\circ(p))(H_0(p)-z)^{-1} = \mathbf{e}(p)\otimes\mathbf{e}(p).$$

Thus, $\lambda_\circ(p)$ is simple and $\mathbf{e}(p)$ is a normalized eigenvector. If $|p| = \rho_c$, $\lambda_\circ(p) = \lambda_c(p)$ and $G(p,k) = \frac{1}{2}(p-k)^2 + \omega(k) - \lambda_c(p) \sim C|k-k(p)|^2$ near $k = k(p)$. It follows that $(H_0(p)-\lambda_\circ(p))\tilde{\mathbf{f}} = 0$ has no solution in $\mathcal{K}$ and $\lambda_\circ(p)$ is not an eigenvalue of $H_0(p)$ if $|p| = \rho_c$. That the half line $[\lambda_c(p),\infty)$ is the absolute continuous spectrum of $H_0(p)$ is a result of the following lemma by virtue of Mourre's theorem ([3]). $\square$

We define $\mathcal{A}$ by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \quad A = \frac{1}{2}\Big((k+h(k)-p)\cdot\frac{\partial}{i\partial k} + \frac{\partial}{i\partial k}\cdot(k+h(k)-p)\Big)$$

where h is a smooth function such that $h(k) = \hat{k}$ when $|k| > \varepsilon$ and $h(k) = 0$ near 0. Since the vector filed $k \to k + h(k) - p$ has bounded derivatives, it generates a global flow $\Phi(t,k)$ on $\mathbf{R}^3$ and $J(t)u(k) = \sqrt{\det \Phi(t,k)}u(\Phi(t,k))$ is a one parameter unitary group on $L^2(\mathbf{R}^3)$. We define $\mathcal{A}$ as the inifinitesimal generator of $J(t)$: $J(t) = e^{it\mathcal{A}}$. We let $\mathcal{D} = C_0^\infty(\mathbf{R}^3)$.

Lemma 3.2. *For any $E \in (\lambda_c(p), \infty) \setminus \{p^2/2\}$, $\mathcal{A}$ is a conjugate operator of $H_0(p)$ at E in the sense of Mourre, viz.*

(1) *$\mathcal{D}$ is a core of both $\mathcal{A}$ and $H_0(p)$.*

(2) *$e^{i\mathcal{A}\alpha}D(H_0(p)) \subset D(H_0(p))$ and $\sup_{|\alpha|<1} \|H_0(p)e^{i\mathcal{A}\alpha}u\| < \infty$ for $u \in D(H_0(p))$.*

(3) *The form $i[H_0(p), \mathcal{A}]$ on $\mathcal{D}$ is bounded from below and closable and the associated selfadjoint operator $i[H_0(p), \mathcal{A}]^0$ satisfies $D(i[H_0(p), \mathcal{A}]^0) \subset D(H_0(p))$.*

(4) *The form defined on $D(\mathcal{A}) \cap D(H_0(p))$ by $[[H_0(p), \mathcal{A}]^0, \mathcal{A}]$ is bounded from $D(H_0(p))$ to $D(H_0(p))^*$.*

(5) *There exists $\alpha > 0$ and $\delta > 0$ and a compact operator K such that*

$$P(E,\delta)i[H_0(p), \mathcal{A}]^0 P(E,\delta) \geq \alpha P(E,\delta) + P(E,\delta)KP(E,\delta)$$

where $P(E,\delta)$ is the spectral projection of $H_0(p)$ for the interval $(E-\delta, E+\delta)$.

Proof. (1) is obvious. Since $D(H_0(p)) = \mathbf{C} \oplus L_2^2(\mathbf{R}^3)$ and $e^{-ct}|k| \leq |\Phi(t,k)| \leq e^{ct}|k|$ for some $c > 0$, (2) is evident also. On $\mathcal{D}$, we compute the commutator

$$i[\mathcal{A}, H_{00}(p)] = \begin{pmatrix} 0 & 0 \\ 0 & (k + h(k) - p) \cdot (k + \hat{k} - p) \end{pmatrix} = L(p). \tag{3.6}$$

Since g_0 is smooth, $[\mathcal{A}, T]$ has a extension to a bounded rank two operator. Thus, $i[\mathcal{A}, H_0(p)]$ is bounded from below, closable and the associated selfadjoint operator has the same domain as $H_0(p)$. This proves (3). (4) holds because $|k|$ has bounded derivatives. If $I \subset (\lambda_c(p), \infty) \setminus \{p^2/2\}$ is a compact interval, then $\frac{1}{2}(p-k)^2 + |k| \in I$ implies that $|k| > \varepsilon$ and

$$(k + h(k) - p) \cdot (k + \hat{k} - p) = (k + \hat{k} - p)^2 > \alpha > 0$$

as, otherwise, there exists a sequence $\{k_j\}$ such that $k_j + \hat{k}_j - p \to 0$ which leads to the contradition that $|p| \geq 1$ and $\frac{1}{2}(p - k_j)^2 + |k_j| \to \lambda_c(\psi)$. Since $i[T, \mathcal{A}]$ is of rank two, (5) follows from the following lemma. $\square$

Lemma 3.3. *Let $\phi \in C_0^\infty(\mathbf{R})$. Then $L(p)\{\phi(H_0(p)) - \phi(H_{00}(p))\}$ is a compact operator.*

Proof. Let $\tilde{\phi}$ be a compactly supported almost analytic extension of ϕ. Then writing $R_0(p) = (H_{00}(p) - z)^{-1}$ and $R(p) = (H_0(p) - z)^{-1}$, we have

$$\phi(H_0(p)) - \phi(H_0(p)) = \frac{1}{2\pi i}\int_{\mathbf{C}} \partial_{\bar{z}}\tilde{\phi}(z) R_0(p,z) T R(p,z) dz \wedge d\bar{z} \tag{3.7}$$

Since $R_0(p,z)$ is a multiplication operator it commutes with $L(p)$ and $L(p)T$ is a compact operator as T is rank two and Image $T \subset D(L(p))$. Since (3.7) is the norm limit of the Riemann sum, the lemma follows. □

3.3 Resolvent and spectrum of H_0

¿From the equations (3.2), (3.3) and (3.4), we derive the formula for the resolvent:

$$(H_0 - z)^{-1}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} G_0(x,z) \\ G_1(x,k,z) \end{pmatrix}. \tag{3.8}$$

Lemma 3.4. *Let $z \notin \mathbf{R}$. Then we have*

$$\hat{G}_0(\xi,z) = \frac{1}{F(\xi,z)}\left(\hat{f}_0(\xi) - \mu\int \frac{g_0(k)\hat{f}_1(\xi-k,k)}{\frac{1}{2}(\xi-k)^2 + \omega(k) - z}dk\right), \tag{3.9}$$

$$\hat{G}_1(\xi,k,z) = \frac{\hat{f}_1(\xi,k)}{\frac{1}{2}\xi^2 + \omega(k) - z} - \frac{\mu g_0(k)\hat{G}_0(\xi+k,z)}{\frac{1}{2}\xi^2 + \omega(k) - z}, \tag{3.10}$$

where $F(\xi,z)$ is given by (2.1).

Since $\lambda_\circ(\rho)$ is strictly inceasing 3.1 implies the following theorem.

Theorem 3.5. *The spectrum of H_0 is absolutely continuous and is given by $\sigma(H_0) = [\lambda_{\min}, \infty)$, where $\lambda_{\min} = \lambda_\circ(0)$.*

4 The behavior of e^{-itH_0}

4.1 Proof of Theorem 1.2

By virtue of Theorem 3.1, $e^{-itH_0(p)}$ can be decomposed as

$$e^{-itH_0(p)} = e^{-it\lambda_\circ(p)}E_p + e^{-itH_0(p)}P_{ac}(H_0(p)),$$

where we set $E_p = 0$ when $|p| \geq \rho_c$. Here $H_0(p)$ is a rank two perturbation of $H_{00}(p)$, $H_{00}(p)$ has a simple eigenvalue $\frac{1}{2}p^2$ and the absolutely continuous spectrum $[\lambda_c(p), \infty)$ and the absolutely continuous subspace $\mathcal{K}_{ac}(H_{00}(p)) = \{0\} \oplus L^2(\mathbf{R}^3)$. It follows by celebrated Kato-Birman's theorem that the limits

$$\lim_{t \to \pm\infty} e^{itH_{00}(p)} e^{-itH_0(p)} P_{ac}(H_0(p)) = \Omega_0^{\pm}(p)$$

exist and the wave operators $\Omega_{00}^{\pm}(p)$ are partial isometries with initial set $\mathcal{K}_{ac}(H_0(p)) = P_{ac}(H_0(p))\mathcal{K}$ onto the final set $\{0\} \oplus L^2(\mathbf{R}^3)$. Thus, as $t \to \pm\infty$, we have for any $\tilde{\mathbf{f}} \in \mathcal{K}$

$$\left\| e^{-itH_0(p)}\tilde{\mathbf{f}} - e^{-it\lambda_\circ(p)} E_p \tilde{\mathbf{f}} - \begin{pmatrix} 0 & 0 \\ 0 & e^{-it(\frac{1}{2}(p-k)^2 + \omega(k))} \end{pmatrix} \Omega_0^{\pm}(p)\tilde{\mathbf{f}} \right\|_{\mathcal{K}} \to 0 \tag{4.1}$$

and $\|\tilde{\mathbf{f}}\|^2 = \|E_p\tilde{\mathbf{f}}\|^2 + \|\Omega_0^{\pm}(p)\tilde{\mathbf{f}}\|^2$. The equation (3.5) implies that $E_pU\mathbf{f}$ has the form

$$E_pU\mathbf{f} = \begin{pmatrix} \hat{f}_{1,0}(p) \\ \hat{f}_{1,1}(p,k) \end{pmatrix} \equiv \begin{pmatrix} \hat{f}_{1,0}(p) \\ \dfrac{\mu g_0(k)\hat{f}_{1,0}(p)}{\frac{1}{2}(p-k)^2 + \omega(k) - \lambda_\circ(p)} \end{pmatrix}$$

with understanding that $\hat{f}_{1,0}(p) = \hat{f}_{1,1}(p,k) = 0$ when $|p| \geq \rho_c$, and

$$U^* \left(\int_{\mathbf{R}^3}^{\oplus} e^{-it\lambda_\circ(p)} E_p dp \right) U\mathbf{f} = \begin{pmatrix} e^{-i\lambda_\circ(D)} f_{1,0}(x) \\ e^{-ikx} e^{-i\lambda_\circ(D_x)} f_{1,1}(x,k) \end{pmatrix}. \tag{4.2}$$

It is obvious that $\hat{f}_{1,0}$ runs over all $L^2(B(\rho_c))$ when f runs over all $\mathcal{H}$ and

$$\left\| U^* \left(\int_{\mathbf{R}^3}^{\oplus} E_p dp \right) U\mathbf{f} \right\|_{\mathcal{H}}^2 = \|f_{1,0}\|_{\mathcal{H}_0}^2 + \|f_{1,1}\|_{\mathcal{H}_1}^2. \tag{4.3}$$

If we write

$$P_{ac} \equiv U^* \left(\int_{\mathbf{R}^3}^{\oplus} \Omega_0^{\pm}(p) dp \right) U, \quad P_{ac}\mathbf{f} = \begin{pmatrix} 0 \\ f_{2,1,\pm} \end{pmatrix},$$

then, P_{ac} is unitary from $U^* \left(\int_{\mathbf{R}^3}^{\oplus} \mathcal{K}_{ac}(H_0(p)) dp \right)$ onto $\{0\} \oplus L^2(\mathbf{R}^6)$ and

$$U^* \left(\int_{\mathbf{R}^3}^{\oplus} \begin{pmatrix} 0 & 0 \\ 0 & e^{-it(\frac{1}{2}(p-k)^2 + \omega(k))} \end{pmatrix} \Omega_0^{\pm}(p) dp \right) U\mathbf{f} = \begin{pmatrix} 0 \\ e^{-it(-\frac{1}{2}\Delta + \omega(k))} f_{2.2,\pm} \end{pmatrix}$$

We insert the relation (4.1) to the identity $e^{-itH_0} = U^* \left(\int_{\mathbf{R}^3}^{\oplus} e^{-itH_0(p)} dp \right) U$ and use the identity (4.2) and (4.4). Theorem 1.2 follows.

4.2 Behavior in configuration space

As the operator $e^{-it(-\frac{1}{2}\Delta+\omega(k))}$ has been well studied, we concentrate on $e^{-i\lambda_\circ(D_x)}v(x)$. When $\hat{v} \in C_0^\infty(B(\rho_c))$, we may apply the method of stationary phase to

$$v(t,x) = \frac{1}{(2\pi)^{3/2}} \int e^{-it\lambda_\circ(\xi)+ix\xi} \hat{v}(\xi) d\xi$$

The points of stationary phase are determined by the equation

$$t\nabla\lambda_\circ(\xi) = x \tag{4.5}$$

which has a unique solution $\xi(x/t)$ when $\frac{x}{t} \in \nabla\lambda_\circ(B(\rho_c))$ due to Lemma 2.2. Thus $v(t,x)$ can be written in the form

$$v(t,x) = \frac{e^{i\phi(t,x)-i\frac{3\pi}{4}}}{t^{3/2}\det(\nabla_\xi^2\lambda_\circ(\xi(x/t)))^{1/2}}(\hat{v}(\xi(x/t)) + t^1 v_1(x/t) + \cdots) \tag{4.6}$$

where the phase function is defined by

$$\phi(t,x) = x \cdot \xi(x/t) - t\lambda_\circ(\xi(x/t)), \tag{4.7}$$

$v_1, v_2, \ldots$ are determined by standard formula and $|v(t,x)| \le C_N t^{-N} \langle x \rangle^{-N}$ for any N for large t.

Lemma 4.1. *Assume $\hat{f}_{1,0} \in C_0^\infty(B(\rho_c))$. Then $e^{-it\lambda_\circ(D)} f_{1,0}(x)$ and Then $e^{-it\lambda_\circ(D_x)} f_{1,1}(x,k)$ has the following asymptotic expansion as $t \to \pm\infty$ for $x \in t\lambda_\circ(B(\rho_c))$:*

$$e^{-it\lambda_\circ(D)} f_{1,0}(x) = \frac{t^{-3/2} e^{i\phi(t,x)-i\frac{3\pi}{4}}}{\det(\nabla_\xi^2\lambda_\circ(\xi(x/t))^{1/2}}(\hat{f}_{1,0}(\xi(x/t)) + t^{-1} g_1(x/t) + \cdots) \tag{4.8}$$

$$e^{-it\lambda_\circ(D)}f_{1,1}(x,k) = \frac{t^{-3/2}e^{i\phi(t,x)-i\frac{3\pi}{4}}}{\det(\nabla_\xi^2\lambda_\circ(\xi(x/t))^{1/2}}\Big(\hat{f}_{1,1}(\xi(x/t),k)$$
$$+t^{-1}M_1(x/t,k)+\cdots\Big), \quad (4.9)$$

where $\phi(t,x)$ *is defined by* (4.7), *using the stationary phase point determined by* (4.5) *and* $g_1(x/t), g_2(x/t), \cdots, M_1(x/t,k), M_2(x/t,k), \ldots$ *are defined by standard formulae involving the derivatives of* $f_{1,0}$ *and* $f_{1,1}$. *For* $x \notin t\lambda_\circ(B(\rho_c))$, *we have for any* N,

$$|e^{-it\lambda_\circ(D)}f_{1,0}(x)| \leq C_N t^{-N}\langle x\rangle^{-N}, \quad (4.10)$$

$$|e^{-it\lambda_\circ(D)}f_{1,1}(x,k)| \leq C_N t^{-N}\langle x\rangle^{-N}\langle k\rangle^{-N}. \quad (4.11)$$

Proof. The formula for $e^{-it\lambda_\circ(D)}f_{1,0}(x)$ is an immediate corollary of (4.6). 4.9 can be proved similarly since $k \longrightarrow \hat{f}_{1,1}(\cdot,k) \in C_0^\infty(B(\rho_c))$ is smooth and rapidly decaying. □

Thus we may consider $e^{-it\lambda_\circ(D)}f_{1,1}(x,k)$ as the part of the wave function, which represents the motion of the electron under the dispersion relation $\lambda_\circ(\xi)$, which is dragging the cloud of photons (however only one photon).

5 Proof of Theorem 1.4

Consider the case that the electron is interacting with the nucleus via the potential V so that the Hamiltonian of the total system is given by

$$H = \begin{pmatrix} -\frac{1}{2}\Delta + V & \mu\langle g| \\ \mu|g\rangle & -\frac{1}{2}\Delta + V + \omega(k) \end{pmatrix}.$$

We introduce the following assumption on the the potential.

Assumption 5.1. *Let* V *be multiplication by a realvalued function* $V(x)$, *such that* V *is bounded relative to* $-\frac{1}{2}\Delta$ *with relative bound less than one. Let* E *be an eigenvalue of* $-\frac{1}{2}\Delta + V$ *with normalized eigenfunction* Ω. *Assume that there exists* $\beta > 5/2$ *such that*

$$|\Omega(x)| \leq C\langle x\rangle^{-\beta}, \quad x \in \mathbf{R}^3. \quad (5.1)$$

We recall that by the Kato-Rellich theorem $-\frac{1}{2}\Delta + V$ is selfadjoint with domain $D(-\frac{1}{2}\Delta + V) = D(-\frac{1}{2}\Delta)$. In many cases we know that the eigenfunction actually has exponential decay, but eigenvalues at a threshold may only decay polynomially.

Associated to each eigenvalue of $-\frac{1}{2}\Delta+V$ satisfying the above assumption is a wave operator, as shown in the following result.

Theorem 5.2. *Let V satisfy Assumption 5.1. For every $f \in L^2(\mathbf{R}^3_k)$, the following limits exist in the strong topology of $\mathcal{H}$:*

$$\lim_{t\to\pm\infty} e^{itH}\begin{pmatrix} 0 \\ e^{-itE-it\omega(k)}\Omega(x)f(k)\end{pmatrix} = W_{\pm}^{E,\Omega}f.$$

Proof. We take $\mu = 1$ to simplify the notation. Since e^{itH} and $L^2(\mathbf{R}^3_k) \ni f \mapsto e^{-itE-it\omega(k)}\Omega \otimes f \in \mathcal{H}$ are isometric operators, it suffices to show that the limits exist for every $f \in C_0^\infty(\mathbf{R}^3 \setminus \{0\})$. For such f the map

$$t \mapsto F_t = e^{itH}\begin{pmatrix} 0 \\ e^{-itE-it\omega(k)}\Omega(x)f(k)\end{pmatrix}$$

is strongly differentiable, and we can easily compute to obtain

$$\frac{d}{dt}F_t = \begin{pmatrix} f_t \\ 0 \end{pmatrix}, \quad f_t = ie^{-itE}\Omega(x)\int_{\mathbf{R}^3} e^{ixk-it\omega(k)}g_0(k)f(k)dk.$$

It suffices to show that $\|f_t\|$ is integrable with respect to t, by the Cook-Kuroda argument. We estimate the integral with respect to k: Since $\nabla_k(xk - t\omega(k)) = x - t\omega(k)^{-1}k$, it follows by integration by parts that outside the set $\{x\colon c|t| < |x| < 2|t|\}$ with c depending on the support of f, we have for any positive N

$$\left|\int_{\mathbf{R}^3} e^{ixk-it\omega(k)}g_0(k)f(k)dk\right| \le C_N|t|^{-N}, \quad |t| \ge 1.$$

The integral is obviously uniformly bounded with respect to (t, x). It follows that

$$\begin{aligned}\int |f_t(x)|^2dx &\le C_N|t|^{-N}\int_{\{x\colon |x|\notin(c|t|,2|t|)\}} |\Omega(x)|^2dx \\ &\quad + C\int_{\{x\colon ct\le|x|\le2|t|\}} |\Omega(x)|^2dx \\ &\le C(|t|^{-N} + \langle t\rangle^{3-2\beta}).\end{aligned}$$

Since $\beta > 5/2$ by Assumption 5.1, it follows that $\|f_t\|$ is integrable. □

6 Proof of Thorem 1.3

We now compare the evolution e^{-itH} to the free evolution e^{-itH_0} and prove Theorem 1.3. By virtue of Theorem 1.2, we have only to prove the following two theorems.

Theorem 6.1. *Let $V(x) = V_1(x) + V_2(x)$ with $V_1 \in L^2(\mathbf{R}^3)$ and $\langle x \rangle^\beta V_2 \in L^\infty(\mathbf{R}^3)$ for some $\beta > 1$. Then, for every $f \in L^2(\mathbf{R}^3_x \times \mathbf{R}^3_k)$, the following limits exist in the strong topology of $\mathcal{H}$:*

$$\lim_{t\to\pm\infty} e^{itH} \begin{pmatrix} 0 \\ e^{it\frac{1}{2}\Delta - it\omega(k)} f(x,k). \end{pmatrix} \tag{6.1}$$

Theorem 6.2. *Let $V \in L^2(\mathbf{R}^3)$. Let $\hat{f}_{1,0} \in L^2(B(\rho_c))$ and $f_{1,1} \in \mathcal{H}_1$ be defined as in* Theorem 1.2, (2). *Then, the following limits exists:*

$$\lim_{t\to\pm\infty} e^{itH} \begin{pmatrix} e^{-i\lambda_0(D_x)} f_{1,0} \\ e^{-ikx} e^{-i\lambda_0(D_x)} f_{1,1} \end{pmatrix}.$$

6.1 Proof of Theorem 6.1

We take $\mu = 1$ to simplify the notation. The set of functions of the form $\sum_{j=1}^N u_j(x) v_j(k)$, with $\hat{u}_j \in C_0^\infty(\mathbf{R}^3_\xi \setminus \{0\})$ and $v_j \in C_0^\infty(\mathbf{R}^3_k)$ is dense in $L^2(\mathbf{R}^3_x \times \mathbf{R}^3_k)$. Thus it suffices to consider $f(x,k) = u(x)v(k)$ with u and v as above. We write again

$$F_t = e^{itH} \begin{pmatrix} 0 \\ e^{it\frac{1}{2}\Delta - it\omega(k)} f(x,k) \end{pmatrix}.$$

We compute the strong derivative with respect to t.

$$\frac{d}{dt} F_t = i \begin{pmatrix} \langle g | e^{it\frac{1}{2}\Delta - it\omega(k)} f \rangle \\ V e^{it\frac{1}{2}\Delta - it\omega(k)} f \end{pmatrix} = \begin{pmatrix} g_{0t}(x) \\ g_{1t}(x,k) \end{pmatrix}.$$

We estimate $g_{1t}(x,k)$ first. We have

$$g_{1t}(x,k) = iV(x)(e^{it\frac{1}{2}\Delta} u)(x) e^{-it\omega(k)} v(k),$$

such that $\|g_{1t}\|_{\mathcal{H}_1} = \|Ve^{it\frac{1}{2}\Delta}u\|_2\|v\|_2$. It follows by the well known estimate for the existence of the wave operator for the two body short potentials (see for example [5]) that $\|g_{1t}\|_{\mathcal{H}_1}$ is integrable with respect to t. The function $g_{0t}(x)$ can be written in the form

$$g_{0t}(x) = i\int g_0(k)e^{ikx-it\omega(k)}v(k)dk\cdot e^{it\frac{1}{2}\Delta}u(x)$$

By Assumption 1.1 and $v \in C_0^\infty$, it follows that the function $w_t(k) = g_0(k)v(k)e^{-it\omega(k)}$ belongs to L^2 with $\|w_t\|_2 = c_0$ independent of t. Thus we can estimate g_{0t} as follows, using the fact that the integral term is the inverse Fourier transform of w_t (up to a constant),

$$\|g_{0t}\|_2 \leq (2\pi)^{3/2}\|\check{w}_t\|_2\|e^{it\frac{1}{2}\Delta}u\|_\infty \leq Cc_0|t|^{-3/2}\|u\|_1.$$

Here we have used the estimate $\|e^{it\frac{1}{2}\Delta}\|_{L^1(\mathbf{R}^3)\to L^\infty(\mathbf{R}^3)} \leq c|t|^{-3/2}$. This estimate shows that $\|g_{0t}\|_2$ is integrable with respect to t such that the limits exist.

6.2 Proof of Theorem 6.2

Since $C_0^\infty(B(\rho_c))$ is dense in $L^2(B(\rho_c))$, it is sufficient to prove the existence of the limits when $\hat{f}_{1,0} \in C_0^\infty(B(\rho_c))$. We use the Cook-Kuroda method. Using the fact that $\lambda_\circ(\xi)$ is the eigenvalue of $H_{00}(\xi)$, it is easy to see that

$$\frac{d}{dt}e^{itH}\begin{pmatrix} e^{-it\lambda_\circ(D_x)}f_{0,1} \\ e^{-it\lambda_\circ(D_x)}f_{1,1}\end{pmatrix} = ie^{itH}\begin{pmatrix} Ve^{-it\lambda_\circ(D_x)}f_{1,0} \\ Ve^{-ikx}e^{-it\lambda_\circ(D_x)}f_{1,1}\end{pmatrix}$$

Thus, it suffices to show that both $\|Ve^{-it\lambda_\circ(D_x)}f_0\|$ and $\|Ve^{-it\lambda_\circ(D_x)}f_1\|$ are intregrable functions of $|t| \geq 1$. But, we have seen in Lemma 4.1 that $|e^{-i\lambda_\circ(D_x)}f_0(x)| \leq Ct^{-3/2}$ and $|e^{-i\lambda_\circ(D_x)}f_1(x,k)| \leq Ct^{-3/2}$. Then, because $V \in L^2$, $\|Ve^{-i\lambda_\circ(D_x)}f_0\| \leq Ct^{-3/2}$ and $\|Vg_0(k)e^{-i\lambda_\circ(D_x)}f_1\| \leq Ct^{-3/2}$ and this completes the proof of the Theorem.

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